数理解析研究所講究録
第2080巻 2018年 139-146

BIFURCATION AND SYMMETRY BREAKING FOR BREZIS-NIRENBERG PROBLEM ON $\mathbb{S}^n$

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1. Introduction

This note presents a brief review of the progressing article [20]. We consider the Brezis-Nirenberg problem on thin annuli in a n-dimensional standard sphere $\mathbb{S}^n$ as an extension of work of Gladiali-Grossi-Pacella-Srikanth [8] that considers the problem on expanding annuli in $\mathbb{R}^n$. We note a similar extension has done by Morabito [15] dealing the problem on expanding annuli in the space which includes a n-dimensional hyperbolic space $\mathbb{H}^n$ as a typical case.

Let $\Delta_{\mathbb{S}^n}$ be a Laplace-Beltrami operator on n-dimensional ($n \geq 2$) sphere $\mathbb{S}^n = \{X = (X_1, \ldots, X_n, X_{n+1}) \in \mathbb{R}^{n+1} : |X| = 1\}$. Further, let $p > 1$ and $\Omega_{\theta_1,\theta_2} = \{X \in \mathbb{S}^n : \cos\theta_1 < X_{n+1} < \cos\theta_2\}$, $\theta_1, \theta_2 \in (0, \pi)$ be a thin annulus on $\mathbb{S}^n$. We consider following Brezis-Nirenberg problem on $\Omega_{\theta_1,\theta_2}$:

$$
\begin{cases} \Delta_{\mathbb{S}^n} U + \lambda U + U^p = 0,\ U > 0 & \text{in } \Omega_{\theta_1,\theta_2}, \\ U = 0 & \text{on } \partial\Omega_{\theta_1,\theta_2}, \end{cases} \tag{1}
$$

Let λ_1 be the first eigenvalue of $-\Delta_{\mathbb{S}^n}$ on $\Omega_{\theta_1,\theta_2}$ and assume $\lambda < \lambda_1$. Moreover let $P : \mathbb{S}^n \setminus \{(0, \ldots, 0, -1)\} \to \mathbb{R}^n$ be a stereographic projection defined by

$$
P(X_1, \ldots, X_n, X_{n+1}) = \frac{1}{X_{n+1} + 1}(X_1, \ldots, X_n), \quad X \in \mathbb{S}^n \setminus \{(0, \ldots, 0, -1)\}. \tag{2}
$$

We define $A_{R,\epsilon} = P(\Omega_{\theta_1,\theta_2})$, concretely

$$
R - \epsilon = \tan\frac{\theta_2}{2}, R + \epsilon = \tan\frac{\theta_1}{2} \quad \text{and} \quad A_{R,\epsilon} = \{x \in \mathbb{R}^n : R - \epsilon < |x| < R + \epsilon\}. \tag{3}
$$

We note on $A_{R,\epsilon}$, Riemannian metric $g = \sum_{i,j} g_{ij} dx_i \otimes dx_j$ is induced, where $g_{ij} = 4\left(1+|x|^2\right)^{-2} \delta_{ij}$. Hence (1) is expressed with this metric as:

$$
(4) \qquad \begin{cases} \Delta w + \dfrac{n(n-2)+4\lambda}{(1+|x|^2)^2} w + 4(1+|x|^2)^{\frac{(n-2)p-(n+2)}{2}} w^p = 0,\ w > 0 & \text{in } A_{R,\epsilon}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad w = 0 & \text{on } \partial A_{R,\epsilon}, \end{cases}
$$

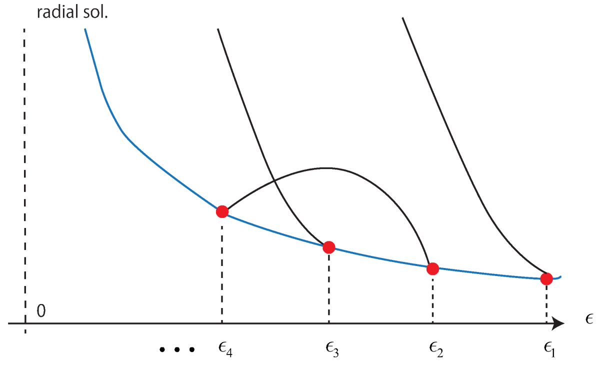
here w satisfies

$$
(5) \qquad U(P^{-1}x) = (1+|x|^2)^{\frac{n-2}{2}} w(x) \quad \text{for } x \in \overline{A_{R,\epsilon}}.
$$

In the next section, regarding ϵ a parameter of $A_{R,\epsilon}$, we obtain results for the existence of bifurcation solutions of (4) from radially symmetric positive solution.

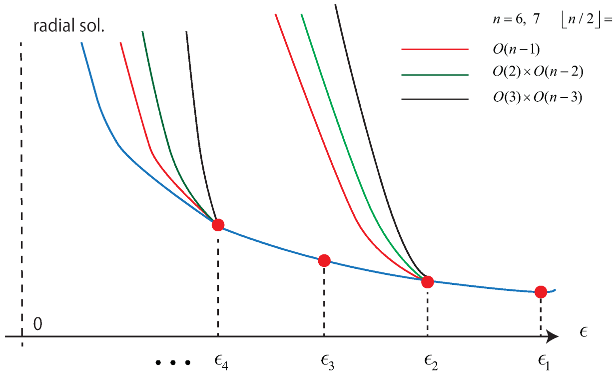
2. Main Results

Theorem 1. *Assume $n \geq 2$, $\lambda < \lambda_1$, $p > 1$. Then there exists $\bar{k} \geq 0$, such that for $k \geq \bar{k}$, we have an unique ϵ_k and at $\epsilon = \epsilon_k$, non-radially symmetric positive bifurcation solution from radially symmetric positive solution of* (4) *exists. Especially, this bifurcation solution has $O(n-1)$ group invariant symmetry. We note ϵ_k satisfies $\lim_{k\to\infty} \epsilon_k = 0$.*



Under the same assumptions with Theorem 1 except that k is even number, we obtain the multiple existence result of bifurcation solutions.

Theorem 2. *Assume $n \geq 2$, $\lambda < \lambda_1$, $p > 1$. Then there exists $\bar{k} \geq 0$, such that for $k \geq \bar{k}$ and k is even number, we have an unique ϵ_k and at $\epsilon = \epsilon_k$, $\lfloor n/2 \rfloor$ non-radially symmetric positive bifurcation solutions from radially symmetric positive solution of* (4) *exists. Especially these bifurcation solutions have $O(h) \times O(n-h)$ $(1 \leq h \leq \lfloor n/2 \rfloor)$ group invariant symmetry respectively. We note ϵ_k satisfies $\lim_{k\to\infty} \epsilon_k = 0$.*



For the proof of Theorems 1 and 2, uniqueness of positive radial solution of (4) plays an important role (we rely the detail, how uniqueness of positive radial solution of (4) is used on [20]). Using the uniqueness result and Leray-Schauder degree argument we obtain Theorems 1 and 2.

3. Uniqueness of the radial positive solution of the equation (4)

We note positive radial solution of (4) satisfies

(6)

$$\begin{cases} w_{rr} + \frac{n-1}{r}w_r + \frac{n(n-2)+4\lambda}{(1+r^2)^2}w + 4(1+r^2)^{\frac{(n-2)p-(n+2)}{2}}w^p = 0, \ w > 0, \ r \in (R-\epsilon, R+\epsilon) \\ w(R\pm\epsilon) = 0. \end{cases}$$

By applying theorems 13 and 14 of [19], we obtain following theorem.

Theorem A . *Let* $\lambda_{n,p} = (6+(6-4n)p)/((p+3)(p-1))$ *and* θ_λ *be a unique* $\theta \in (0, \pi/2)$ *satisfying* $G(\tan(\theta/2)) = 0$ *where*

$$G(r) = Ar^4 + Br^2 + A,$$

and

$$\begin{aligned} A &= \big(n+2-(n-2)p\big)\big((n-2)p+n-4\big) \\ &= (p+3)[3n^2-6n-(n^2-4n+4)p] - 8(n-1)^2, \\ B &= (p+3)[-6n^2+12n+(2n^2+4\lambda-4)p+2\lambda p^2-6\lambda-12] + 16(n-1)^2. \end{aligned}$$

Moreover, assume $\epsilon > 0$ *be small enough. Then, the equation* (6) *has a unique positive solution for* $1 < p$ *and* $\lambda \in (-\infty, \lambda_{1,\epsilon})$ *except for the following four cases.*

(i) $n \geq 3,\ 1 < p \leq (n+2)/(n-2),\ \lambda \in (-\infty, \lambda_{n,p})$ *and* $1 \in (R-\epsilon, R+\epsilon)$.
(ii) $n \geq 3,\ p > (n+2)/(n-2),\ \lambda \in (-\infty, \lambda_{n,p}]$ *and* $1 \in (R-\epsilon, R+\epsilon)$.
(iii) $n = 2,\ 1 < p,\ \lambda \in (-\infty, -2/(p+3))$ *and* $1 \in (R-\epsilon, R+\epsilon)$.

(iv) $n \geq 3$, $p > (n+2)/(n-2)$, $\lambda \in (\lambda_{n,p}, \lambda_{1,\epsilon})$ *and* $\tan(\theta_\lambda/2) \in (R-\epsilon, R+\epsilon)$ *or* $\tan((\pi - \theta_\lambda)/2) \in (R-\epsilon, R+\epsilon)$.

On the other hand, applying Theorem 2.21 and Theorem 2.24 of Ni-Nussbaum [17], we have the following result.

Theorem B . *Let* $\lambda \geq -n(n-2)/4$, $n \geq 2$ *and*

$$\left(\frac{R+\epsilon}{R-\epsilon}\right) \leq \begin{cases} (n-1)^{\frac{1}{n-2}}, & n \geq 3 \\ e, & n = 2. \end{cases} \tag{7}$$

Then equation (6) *has a unique positive solution for* $1 < p$ *and* $\lambda \in (-\infty, \lambda_{1,\epsilon})$.

Remark 1. *Since we assume* $\epsilon > 0$ *is small, we can remove the assumption* (7). *Nevertheless, even combining Theorem A and Theorem B, following three cases remain that do not guarantee the uniqueness of the positive solution of* (6) *(note that in the case* $p > (n+2)/(n-2)$, $\lambda_{n,p} > -n(n-2)/4$ *holds).*

(I) $n \geq 3$, $1 < p \leq (n+2)/(n-2)$, $\lambda \in (-\infty, \lambda_{n,p})$ *and* $1 \in (R-\epsilon, R+\epsilon)$.

(II) $n \geq 3$, $p > (n+2)/(n-2)$, $\lambda \in (-\infty, -n(n-2)/4)$ *and* $1 \in (R-\epsilon, R+\epsilon)$.

(III) $n = 2$, $1 < p$, $\lambda \in (-\infty, -2/(p+3))$ *and* $1 \in (R-\epsilon, R+\epsilon)$.

We also note that in the case of Brezis-Nirenberg problem on thin annulus with Dirichlet boundary condition in $\mathbb{R}^n$, *uniqueness of positive radial solution without any restriction can be obtained through Theorem 7 of* [19] *and Theorem 2.21 of* [17]. *This difference with* $\mathbb{S}^n$ *case motivates the study of progressing article* [20].

Assume p satisfies $1 < p$. We consider (6) in somewhat generalized form:

$$\begin{cases} u_{rr} + \dfrac{f_r(r)}{f(r)} u_r - g(r)u + h(r)u^p = 0, \ u > 0, \quad r \in (R', R''), \\ u(R') = 0, \ u(R'') = 0, \end{cases} \tag{8}$$

where $-\infty < R' < R''$, $f \in C^1([R', R''])$ and f is positive and non-decreasing on (R', R''), $g \in C^1((R', R'')) \cap C([R', R''])$, $h \in C^1((R', R'')) \cap C([R', R''])$ and h is positive on $[R', R'')$. In the case $R'' = \infty$, $u(R'') = 0$ means $\lim_{r \to \infty} u(r) = 0$.

We consider the uniqueness of positive solution of (8). We can show the following lemma.

Lemma 1. *Let* u_1 *and* u_2 *be solutions of* (8) *satisfying* $u_{1,r}(R') > u_{2,r}(R')$. *Then it holds that*

$$\frac{d}{dr}\left(\frac{u_1(r)}{u_2(r)}\right) > 0, \quad r \in (R', R). \tag{9}$$

3.1. **An application to equation** (6).

First, we introduce an auxiliary function φ as

$$
\text{(10)} \qquad \begin{cases} \left(r^{n-1}\varphi_r(r)\right)_r + \frac{n(n-2)+4\lambda}{(1+r^2)^2} r^{n-1}\varphi(r) = 0 \text{ in } r \in (R-\epsilon_0, R+\epsilon_0) \\ \varphi(R-\epsilon_0) = 1,\ \varphi_r(R-\epsilon_0) = 1 \\ \varphi \text{ is monotone increasing on } r \in (R-\epsilon_0, R+\epsilon_0), \end{cases}
$$

where ϵ_0 is a small positive number. We note if $\epsilon_0 > 0$ is sufficiently small, φ satisfying the above monotone property clearly exists. Here we put

$$
\text{(11)} \qquad w(r) = \varphi(r)u(r)
$$

then (6) with $0 < \epsilon < \epsilon_0$ can be rewritten as

(12)

$$
\begin{cases} \left(r^{n-1}\varphi(r)^2 u_r(r)\right)_r + 4r^{n-1}\left(1+r^2\right)^{\frac{(n-2)p-(n+2)}{2}} \varphi(r)^{p+1}u(r)^p = 0, & r \in (R-\epsilon, R+\epsilon), \\ u(r) > 0, & r \in (R-\epsilon, R+\epsilon), \\ u(R \pm \epsilon) = 0. \end{cases}
$$

Hence putting $R' = R - \epsilon, R'' = R + \epsilon$ and

$$
\text{(13)} \qquad \begin{cases} f(r) = r^{n-1}\varphi(r)^2 \\ g(r) \equiv 0 \\ h(r) = 4\left(1+r^2\right)^{\frac{(n-2)p-(n+2)}{2}} \varphi(r)^{p-1}, \end{cases}
$$

we see that equation (12) takes the form of (8).

Remark 2. *We note above f, g, h satisfies the properties assumed in* (8). *Especially, non-decreasing property of $f(r)$, $r \in (R', R'')$ holds.*

Now we introduce Pohožaev function.

Definition 1. *For positive solutions u of* (8) *with f, g, h as* (13) *and a, b, c of class $C^1[R-\epsilon, R+\epsilon]$ functions, we define Pohožaev function $J(r;u)$ as*

$$
\text{(14)} \qquad J(r;u) = \frac{1}{2}a(r)u_r(r)^2 + b(r)u_r(r)u(r) + \frac{1}{2}c(r)u(r)^2 + \frac{1}{p+1}a(r)h(r)u(r)^{p+1}.
$$

For such $J(r;u)$, we obtain by direct computation that

$$
\frac{d}{dr}J(r;u) = A(r)u_r(r)^2 + B(r)u_r u(r) + G(r)u(r)^2 + H(r)u(r)^{p+1},
$$

where

$$
\begin{cases}
A(r) = \frac{1}{2}a_r(r) - \frac{f_r(r)}{f(r)}a(r) + b(r) \\
B(r) = b_r(r) - \frac{f_r(r)}{f(r)}b(r) + c(r) \\
G(r) = \frac{1}{2}c_r(r) \\
H(r) = -b(r)h(r) + \frac{1}{p+1}\left(a(r)h(r)\right)_r.
\end{cases} \tag{15}
$$

Here we define $F_1(r)$ and $F_2(r)$ as

$$
F_1(r) = \int_R^r \frac{dt}{f(t)} \quad \text{and} \quad F_2(r) = \int_R^r \frac{F_1(t)}{f(t)}dt, \tag{16}
$$

respectively. We put

$$
\begin{cases}
c(r) & = -1 \\
b(r) & = c_1 f(r) + f(r)F_1(r) \\
a(r) & = c_2 f(r)^2 - 2c_1 f(r)^2 F_1(r) - 2f(r)^2 F_2(r),
\end{cases} \tag{17}
$$

where c_1 and c_2 are arbitrary real constant. Then we can easily see that $A(r) \equiv B(r) \equiv G(r) \equiv 0$ on $[R-\epsilon, R+\epsilon]$ and hence

$$
\frac{dJ(r;u)}{dr} = H(r)u(r)^{p+1}. \tag{18}
$$

Now, we fix the constant c_1 and c_2 as

$$
\begin{cases}
c_1(\epsilon) = \dfrac{F_2(R-\epsilon) - F_2(R+\epsilon)}{F_1(R+\epsilon) - F_1(R-\epsilon)} \\
c_2(\epsilon) = \dfrac{2\left(F_1(R+\epsilon)F_2(R-\epsilon) - F_1(R-\epsilon)F_2(R+\epsilon)\right)}{F_1(R+\epsilon) - F_1(R-\epsilon)}.
\end{cases} \tag{19}
$$

Then we can see that

$$
a(R \pm \epsilon) = 0 \tag{20}
$$

holds.

Remark 3. *In* [18,19], *$a(r), b(r)$ and $c(r)$ are taken to satisfy $A(r) \equiv B(r) \equiv H(r) \equiv 0$. Hence, in* [18,19], *Pohožaev function satisfies*

$$
\frac{dJ(r;u)}{dr} = G(r)u(r)^2.
$$

Next, we show that $a(r)$ and $b(r)$ are of order $O(\epsilon)$.

Lemma 2. *Let $a(r)$ and $b(r)$ be as* (17), *further $c_1(\epsilon)$ and $c_2(\epsilon)$ be as* (19). *Then, it holds that*

$$
|a(r)| \le C_1\epsilon, \quad |b(r)| \le C_2\epsilon, \quad r \in (R-\epsilon, R+\epsilon)
$$

where C_1 and C_2 are positive constants independent of ϵ.

Using this lemma, we can show the monotone property of H.

Lemma 3. *For sufficiently small $\epsilon > 0$, $H(r)$ is monotone decreasing on $(R-\epsilon, R+\epsilon)$ and it holds that $H(R-\epsilon) > 0$, $H(R+\epsilon) < 0$.*

Using these lemmas, we can prove the uniqueness of the positive solution of (12) without assuming (I)-(III) of Remark 1.

Theorem 3. *Let $\epsilon > 0$ be sufficiently small. Then, the equation (6) has a unique positive solution for $1 < p$ and $\lambda \in (-\infty, \lambda_{1,\epsilon})$.*

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